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Coherent states for polynomial $su(2)$ algebra

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Abstract
A class of generalized coherent states is constructed for a polynomial $su(2)$ algebra in a group-free manner. As a special case, the coherent states for the cubic $su(2)$ algebra are discussed. The states so constructed reduce to the usual $SU(2)$ coherent states in the linear limit.

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1. Introduction

In the present paper, we construct a class of coherent states for a polynomial $su(2)$ algebra by minimally generalizing the usual $SU(2)$ coherent states. The polynomial $su(2)$ algebra is a deformed algebra whose generators obey the following relations,

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm, \qquad [\hat{J}_+, \hat{J}_-] = \Psi(\hat{J}_0) \tag{1}$$

where $\Psi(\hat{J}_0)$ is a polynomial in $\hat{J}_0$. This algebra accommodates the quadratic and the cubic algebra as special cases. The cubic algebra was first considered by Higgs [1] and by Leeman [2] in dealing with the harmonic oscillator and the Kepler problem on a two-dimensional sphere, while the quadratic algebra was first analyzed by Sklyanin [3] in conjunction with the quantum group. The cubic algebra, in particular, has appeared in various areas of study including the identical particle symmetry in two dimensions [4], the Calogero model [5], multiphoton processes [6, 8], quantum dot problems [7] and others. In recent years, considerable attention has been given to the construction of coherent states for such nonlinearly deformed algebra. In [9], Cannata, Junker and Trost constructed coherent states for the quadratic $su(1, 1)$ algebra stemming from supersymmetric quantum mechanics by demanding them to be eigenstates of the noncompact operator in much the same way that Barut and Girardello [10] constructed the $SU(1, 1)$ coherent states. In [8], Sunilkumar *et al* proposed a general framework for finding coherent states of polynomially deformed algebras including the quadratic and cubic algebras, and used the procedure to construct the polynomially deformed $su(1, 1)$ coherent states for quantum optics.

In this paper, we first construct a class of Perelomov-like coherent states for a nonlinearly deformed $su(2)$ algebra of Bonatos, Danskaloyannis and Kolokotronis [13]. Since an analogue of the usual exponential map from $su(2)$ to $SU(2)$ can hardly be found, Perelomov's group-theoretic procedure [11] is not immediately applicable in construction of the coherent states for the nonlinearly deformed $su(2)$ algebra. The approaches taken in [9, 8] are also unsuited to our purposes. Thus, giving up the group-theoretic procedure as Klauder [14] advocated in constructing the hydrogen atom coherent states, we generalize minimally the usual $SU(2)$ coherent states [12]. Then we choose the structure function of the algebra so that the deformed algebra is specified to be a polynomial $su(2)$ algebra of odd degree $2p - 1$ which includes the cubic $su(2)$ coherent states ($p = 2$) as a special case. We also show that the cubic $SU(2)$ coherent states reduce smoothly to the usual $SU(2)$ coherent states when an appropriate linear limit is taken.

2. Polynomial $su(2)$ algebra

In [13], Bonatsos, Daskaloyannis and Kolokotronis proposed a deformed $su(2)$ algebra, denoted by $su_\Phi(2)$, which has representations similar to those of the usual $su(2)$. In their deformation, the three generators $\{\hat{J}_0, \hat{J}_+, \hat{J}_-\}$ of the algebra obey

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm, \qquad [\hat{J}_+, \hat{J}_-] = \Phi(\hat{J}_0(\hat{J}_0 + 1)) - \Phi(\hat{J}_0(\hat{J}_0 - 1)). \tag{2}$$

It is important to assume that the structure function $\Phi(x)$ is an increasing function of x defined for $x > -1/4$. If x is an operator, it is operator-valued. The Casimir operator of $su_\Phi(2)$ is

$$\hat{C} = \hat{J}_-\hat{J}_+ + \Phi(\hat{J}_0(\hat{J}_0 + 1)) = \hat{J}_+\hat{J}_- + \Phi(\hat{J}_0(\hat{J}_0 - 1)). \tag{3}$$

On the basis $\{|j, m\rangle\}$ that diagonalizes $\hat{J}_0$ and $\hat{C}$ simultaneously such that

$$\hat{C}|j, m\rangle = \Phi(j(j + 1))|j, m\rangle \qquad \hat{J}_0|j, m\rangle = m|j, m\rangle, \tag{4}$$

we have

$$\hat{J}_+|j, m\rangle = \sqrt{\Phi(j(j + 1)) - \Phi(m(m + 1))}|j,\ m + 1\rangle \tag{5}$$

$$\hat{J}_-|j, m\rangle = \sqrt{\Phi(j(j + 1)) - \Phi(m(m - 1))}|j,\ m - 1\rangle \tag{6}$$

with

$$2j = 0, 1, 2, \ldots, \qquad |m| \leqslant j. \tag{7}$$

In the present paper, we consider a special case of $su_\Phi(2)$ with a structure function given by a homogeneous polynomial of degree p,

$$\Phi(x) = \sum_{r=1}^{p} \alpha_r x^r \qquad (\alpha_1 > 0, \alpha_p \neq 0) \tag{8}$$

where α_r are real constants. Since the structure function $\Phi(x)$ when acting on the state $|j, m\rangle$ is required to be an increasing function of $x = j(j + 1)$, the following condition must be satisfied,

$$\sum_{r=1}^{p} r\alpha_r[j(j + 1)]^{r-1} > 0. \tag{9}$$

Here we must note that if $\alpha_p < 0$ then j has a maximum value $j_{\max}$. This implies that the representation space becomes finite dimensional for a given negative value of α_p.

Substitution of (8) into (2) leads to the polynomial $su(2)$ algebra of odd degree $2p - 1 (p = 1, 2, 3, \ldots)$; namely,

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm, \qquad [\hat{J}_+, \hat{J}_-] = 2\sum_{r=1}^{p} \alpha_r \hat{J}_0^r \sum_{s=1}^{r} (\hat{J}_0 + 1)^{r-s}(\hat{J}_0 - 1)^{s-1}. \tag{10}$$

which we denote by $su_{2p-1}(2)$. Here, $p = 1$ and $p = 2$ correspond to the usual $su(2)$ and the cubic $su(2)$ case, respectively; that is, $su_1(2) = su(2)$ and $su_3(2) = su_{cub}(2)$. Note that if the structure function is chosen to be a polynomial in $x = \hat{J}_0(\hat{J}_0 + 1)$ of degree p, then the deformation is given by a polynomial in $\hat{J}_0$ of degree $2p - 1$. Therefore, the quadratic algebra cannot be derived from $su_\Phi(2)$.

3. Coherent states for polynomial $su(2)$

In constructing coherent states for a deformed $su(2)$ algebra, the standard group theoretical method is not immediately applicable because of the lack of the corresponding Lie group. Since the polynomial $su(2)$ algebra (10) reduces to the usual $su(2)$ when $p = 1$, we adopt a simple guiding principle that the set of the constructed coherent states for $su_{2p-1}(2)$ will reduce to the usual set of $SU(2)$ coherent states in the linear limit ($p = 1$).

Coherent states for $su_\Phi(2)$. First let us construct a set of coherent states for the deformed algebra $su_\Phi(2)$. As is in the case of $su(2)$, the lowest state $|j, -j\rangle (m = -j)$ is taken as the fiducial state:

$$\hat{J}_-|j, -j\rangle = 0. \tag{11}$$

Operating on the fiducial state with the generator $\hat{J}_+$ of the deformed algebra, we construct the following states,

$$|j, \xi\rangle = N_\Phi^{-1}(|\xi|)e^{\xi \hat{J}_+}|j, -j\rangle, \tag{12}$$

where $N(|\xi|)$ is the normalization factor and $\xi \in \mathbf{C}$. These states are similar in form to the usual $SU(2)$ coherent states. However, here $\hat{J}_+$ is an operator satisfying the deformed algebra $su_\Phi(2)$ rather than the linear $su(2)$ algebra. Thus $e^{\xi \hat{J}_+}$ is not meant to be a representative of the coset space associated with the usual $SU(2)$ group since no Lie group can be formed by the exponential map of the deformed algebra.

As is evident from (5) that $\hat{J}_+|j, j\rangle = 0$, the states (12) can be expressed as

$$|j, \xi\rangle = N_\Phi^{-1}(|\xi|)\sum_{n=0}^{2j}\frac{\xi^n \hat{J}_+^n}{n!}|j, -j\rangle. \tag{13}$$

Again from (5) it follows for $0 \leqslant n \leqslant 2j (-j \leqslant m \leqslant j)$ that

$$\hat{J}_+^n|j, -j\rangle = \sqrt{[k_n]!}|j, -j + n\rangle \tag{14}$$

where

$$k_n = \Phi(j(j + 1)) - \Phi((j - n)(j - n + 1)). \tag{15}$$

In the above, we have used the factorial notation of k_n to signify

$$[k_n]! = \prod_{l=1}^{n} k_l. \tag{16}$$

Hence the states (13) take the form,

$$|j, \xi\rangle = N_\Phi^{-1}(|\xi|)\sum_{n=0}^{2j}\frac{\sqrt{[k_n]!}}{n!}\xi^n|j, -j + n\rangle \tag{17}$$

or

$$|j, \xi\rangle = N_\Phi^{-1}(|\xi|)\sum_{m=-j}^{j}\frac{\sqrt{[k_{j+m}]!}}{(m + j)!}\xi^{j+m}|j, m\rangle. \tag{18}$$

The normalization factor is determined by

$$N_\Phi^2(|\xi|) = \sum_{n=0}^{2j} \frac{[k_n]!|\xi|^{2n}}{(n!)^2}. \tag{19}$$

As has been assumed, the structure function $\Phi(x)$ is an increasing function of x. Hence $k_n \geqslant 0$ for $0 \leqslant n \leqslant 2j$. Thus $N_\Phi^2(|\xi|)$ has no zeros and the states (17) are all normalized to unity. Furthermore, from Schwarz's inequality, it is obvious that the inner product of two states obeys

$$\langle j, \xi | j, \xi' \rangle = [N_\Phi(|\xi|) N_\Phi(|\xi'|)]^{-1} \sum_{n=0}^{2j} \frac{[k_n]!(\xi^* \xi')^n}{(n!)^2} \leqslant 1. \tag{20}$$

The inner product is not generally zero even when $\xi \neq \xi'$.

The states (17) may admit the resolution of unity

$$\int |j, \xi\rangle \, \mathrm{d}\mu_\Phi(\xi, \xi^*) \langle j, \xi| = 1 \tag{21}$$

provided that the following measure can be found,

$$\mathrm{d}\mu_\Phi(\xi, \xi^*) = \frac{1}{2\pi} N_\Phi^2(|\xi|) \rho_\Phi(|\xi|^2) \, \mathrm{d}|\xi|^2 \, \mathrm{d}\phi. \tag{22}$$

Here we let $\xi = |\xi| \, \mathrm{e}^{\mathrm{i}\phi} (0 \leqslant \phi < 2\pi)$, and seek a weight function $\rho_\Phi(|\xi|^2)$ satisfying

$$\int_0^\infty \rho_\Phi(t) t^n \, \mathrm{d}t = \frac{(n!)^2}{[k_n]!}. \tag{23}$$

In this regard, we consider the set of states constructed above as a formal set of coherent states for $su_\Phi(2)$ with a structure function $\Phi(x)$ unspecified. To make them as those for the polynomial algebra, we have to calculate $[k_n]!$ explicitly for the chosen structure function (8).

Coherent states for $su_{2p-1}(2)$. Substitution of the structure function (8) into (15) yields

$$k_n = n(2j + 1 - n)\chi_n \tag{24}$$

where

$$\chi_n = \sum_{r=1}^{p} \sum_{s=1}^{r} \alpha_r [j(j+1)]^{r-s} [(j-n)(j-n+1)]^{s-1}. \tag{25}$$

The generalized factorial of $(2j + 1 - n)$ signifies

$$[2j - n + 1]! = (2j)(2j-1)(2j-2) \cdots (2j - n + 1) = \frac{(2j)!}{(2j-n)!} = n! \binom{2j}{n}. \tag{26}$$

Thus the states (17) can be cast into the form,

$$|j, \xi\rangle = N_p^{-1}(|\xi|) \sum_{n=0}^{2j} \binom{2j}{n}^{1/2} \sqrt{[\chi_n]!} \xi^n |j, -j + n\rangle \tag{27}$$

with the normalization,

$$N_p^2(|\xi|) = \sum_{n=0}^{2j} \binom{2j}{n} [\chi_n]! |\xi|^{2n}. \tag{28}$$

The set of states thus obtained in (27) with the factor χ_n specified by (25) is indeed a set of coherent states for $su_{2p-1}(2)$. Apparently the usual $SU(2)$ coherent states are obtained from (27) if $\chi_n = 1$ for all n. Hence χ_n is the very factor that characterizes the nonlinear deformation of the $SU(2)$ coherent states. Here it will be referred to as the deformation factor or the χ factor in short.

The deformation factor χ_n is an inhomogeneous polynomial in n of degree $2p-2$, which may be factorized, with $\alpha_p \neq 0$, in the form,

$$\chi_n = \alpha_p(n-a_1)(n-a_2)\cdots(n-a_{2p-2}) \tag{29}$$

where a_i's are the roots of $\chi_n = 0$ with respect to n. Accordingly, the generalized factorial of χ_n is given by

$$[\chi_n]! = \chi_1\chi_2\cdots\chi_n = \alpha_p^n \prod_{i=1}^{2p-2}(1-a_i)_n \tag{30}$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad (a)_0 = 1.$$

The factorial $[2j-n+1]!$ given in (26) may also be put in an alternative form,

$$[2j-1+n]! = (2j)(2j-1)(2j-2)\cdots(2j-n+1) = (-1)^n(-2j)_n. \tag{31}$$

As a result, the factor $[k_n]!$ becomes

$$[k_n]! = (-1)^n\alpha_p^n n!(-2j)_n(1-a_1)_n(1-a_2)_n\cdots(1-a_{2p-2})_n. \tag{32}$$

With this, the normalization factor (19) for the polynomial algebra is expressed in terms of Pochhammer's generalized hypergeometric function,

$$N_p^2(|\xi|) = {}_{2p-1}F_0(-2j, 1-a_1, 1-a_2, \ldots, 1-a_{2p-2}; -\alpha_p|\xi|^2), \tag{33}$$

which is of course a polynomial in $|\xi|^2$ of degree $2j$ as $(-2j)_n = \Gamma(-2j+n)/\Gamma(-2j) = 0$ for $n > 2j$. In this way the normalization of each coherent state is given in closed form. Similarly, the inner product of two distinct coherent states is given by

$$\langle j,\xi|j,\xi'\rangle = N_p^{-2}(|\xi|)\,{}_{2p-1}F_0(-2j, 1-a_1, 1-a_2, \ldots, 1-a_{2p-2}; -\alpha_p\xi^*\xi') \tag{34}$$

which is not generally zero for $\xi \neq \xi'$. The set of these states is overcomplete.

Resolution of unity. The coherent states thus constructed for $su_{2p-1}(2)$ are able to resolve unity as

$$\int |j,\xi\rangle\, \mathrm{d}\mu_p(\xi,\xi^*)\langle j,\xi| = 1 \tag{35}$$

with the measure,

$$\mathrm{d}\mu_p(\xi,\xi^*) = \frac{1}{2\pi}N_p^2(|\xi|)\rho_p(|\xi|^2)\,\mathrm{d}|\xi|^2\,\mathrm{d}\phi. \tag{36}$$

The weight function $\rho_p(|\xi|^2)$ is determined by (23). From (26) and (30) follows

$$[k_n]! = \frac{\alpha_p^n(2j)!\Gamma(n+1)}{\Gamma(2j+1-n)}\prod_{i=1}^{2p-2}\frac{\Gamma(n+1-a_i)}{\Gamma(1-a_i)}. \tag{37}$$

Substitution of this into (30) leads to

$$\int_0^\infty \rho_p(t)t^n\,\mathrm{d}t = \frac{\alpha_p^{-n}\prod_{i=1}^{2p-2}\Gamma(1-a_i)}{(2j)!}\frac{\Gamma(n+1)\Gamma(n+1-a_i)}{\prod_{i=1}^{2p-2}\Gamma(n+1-a_i)}. \tag{38}$$

It turns out that the weight function is given in terms of Meijer's G-function (see Formula 7.811.4 in [15]) as

$$\rho_p(|\xi|) = \frac{\alpha_p\prod_{i=1}^{2p-2}\Gamma(1-a_i)}{(2j)!}G^{1\,1}_{2p-2\,1}\left(\alpha_p|\xi|^2\,\middle|\,\begin{matrix}-(2j+1), -a_1, -a_2, \ldots - a_{2p-2}\\ 0\end{matrix}\right). \tag{39}$$

With the weight function (39) for the measure (36), the resolution of unity (35) can indeed be achieved.

4. Coherent states for the cubic algebra

The algebra considered by Higgs for the harmonic oscillator on a sphere S^2 is cubic; namely, its generators obey the commutation relations,

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm, \qquad [\hat{J}_+, \hat{J}_-] = 2\alpha\hat{J}_0 - 4\beta\hat{J}_0^3, \tag{40}$$

where α and β are positive constants. Certainly this is a special case of the polynomial algebra (10) with $p = 2, \alpha_1 = \alpha$ and $\alpha_2 = -\beta$. The corresponding structure function is quadratic in x,

$$\Phi(x) = \alpha x - \beta x^2. \tag{41}$$

In order for this $\Phi(x)$ to remain as an increasing function, it is necessary to meet the condition, $x = j(j+1) < \alpha/(2\beta)$. This condition implies that $2j$ of (7) has a maximum value $2j_{\max} < \sqrt{(2\alpha/\beta)+1} - 1$ for each fixed value of α/β. The cubic nature (41) is contained only in the deformation factor; namely,

$$\chi_n = \alpha - \beta[n^2 - (2j+1)n + 2j(j+1)], \tag{42}$$

which is quadratic in n. Factorizing the χ-factor in the form

$$\chi_n = -\beta(n-a)(n-b) \tag{43}$$

with the zeros,

$$a, b = \tfrac{1}{2}[(2j+1) \pm \sqrt{(2j+1)^2 - 8j(j+1) + 4\alpha/\beta}]. \tag{44}$$

Note that the roots a and b are j-dependent. From (27) and (43) immediately follow the coherent states for the cubic $su(2)$ of the form,

$$|j, \xi\rangle = N_2^{-1}(|\xi|) \sum_{n=0}^{2j} \binom{2j}{n}^{1/2} [(-\beta)^n (1-a)_n (1-b)_n]^{1/2} \xi^n |j, -j+n\rangle. \tag{45}$$

With (43) the normalization factor (28) becomes

$$N_2^2(|\xi|) = {}_3F_0(-2j, 1-a, 1-b; \beta|\xi|^2). \tag{46}$$

The resolution of unity is achieved with the weight function,

$$\rho_2(|\xi|^2) = -\beta \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(2j+1)} G_{3\,1}^{1\,1}\left(-\beta|\xi|^2 \,\middle|\, \begin{matrix} -(2j+1), -a, -b \\ 0 \end{matrix}\right). \tag{47}$$

Next we wish to show that the cubic $SU(2)$ coherent states (45) with $\alpha = 1$ reduce to the usual SU(2) coherent states in the limit $\beta \to 0$. For small β the roots (44) of $\chi_n = 0$ behave as

$$a \sim \frac{1}{\sqrt{\beta}}, \qquad b \sim -\frac{1}{\sqrt{\beta}}. \tag{48}$$

Thus, in the limit $\beta \to 0$, $[\chi_n]! \to 1$ and

$$\lim_{\beta\to 0} N_2^2(|\xi|) = {}_1F_0(-2j; -|\xi|^2) = (1+|\xi|^2)^{2j}. \tag{49}$$

The cubic coherent states (45) with $\alpha = 1$ reduce to

$$|j, \xi\rangle = (1+|\xi|^2)^{-j} \sum_{n=0}^{2j} \binom{2j}{n}^{1/2} \xi^n |j, -j+n\rangle \tag{50}$$

which are the standard normalized $SU(2)$ coherent states. By the same limiting procedure, the weight function $\rho_2(|\xi|^2)$ of (47) goes (via Formula 9.348 in [15]) to

$$\rho_1(|\xi|^2) = \frac{1}{\Gamma(2j+1)} G^{1\,1}_{1\,1}\left(|\xi|^2 \middle| \begin{matrix} -(2j+1) \\ 0 \end{matrix}\right) = (2j+1)_1F_0(2j+2; -|\xi|^2) \tag{51}$$

or

$$\rho_1(|\xi|^2) = (2j+1)(1+|\xi|^2)^{-2j-2}. \tag{52}$$

5. Concluding remarks

We have constructed a set of coherent states for a polynomial $su(2)$ algebra based on the nonlinear algebra $su_\Phi(2)$ of Bonatsos, Daskaloyannis and Kolokotronis. Our discussion has been limited to the polynomial deformation of odd degree. If the structure function $\Phi(x)$ is a polynomial in $x = \hat{J}_0(\hat{J}_0+1)$ (of either even or odd degree), the algebra is always of a polynomial in $\hat{J}_0$ of odd degree. Thus the constructed coherent states contain the usual $SU(2)$ states and the cubic $SU(2)$ states, but preclude the quadratic $SU(2)$ states. A different approach is needed to construct coherent states for a polynomial algebra of even degree. If $\alpha_1 = -1$, then the polynomial algebra becomes a nonlinear deformation of $su(1,1)$, whose coherent states will be discussed elsewhere.

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